Stable Adams operations on $RO(C_2)$-graded homotopy groups

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Abstract

The C_2-spectrum of Atiyah's Real K-theory is denoted by $\mathbf{KR}$ and the C_2-spectrum of topological modular forms of level structure $\Gamma_1(3)$ by $\mathbf{TMF}_1(3)$. In this short note we compute the C_2-equivariant stable Adams operations on the $RO(C_2)$-graded homotopy groups of $\mathbf{KR}$ and $\mathbf{TMF}_1(3)$.

1 Introduction

In the study of the connection between elliptic curves and cohomology theories, Hopkins [Hop95] constructed the universal elliptic cohomology theory TMF of topological modular forms. This is the spectrum corresponding to the moduli stack $\mathcal{M}_{ell}$ of elliptic curves. Analog to the situation in classical modular forms, there exist spectra of topological modular forms of certain level structures. The moduli stack $\mathcal{M}_0(n)$, classifying elliptic curves with a chosen cyclic subgroup of order n, yields the spectrum $\mathrm{TMF}_0(n)$, and the moduli stack $\mathcal{M}_1(n)$, which classifies elliptic curves with a chosen point of exact order n, has associated spectrum $\mathrm{TMF}_1(n)$.

Moreover, Adams operations on K-theory proved to be important tools in the past. Two of the major uses were Adams and Atiyah's [AA66] proof of the Hopf invariant one problem using unstable Adams operations and Quillen's [Qui72] computation of the algebraic K-theory of finite fields using stable Adams operations. As topological modular forms are to be viewed as a higher chromatic analogue of topological K-theory, it is natural to ask for the existence of stable Adams operations on the different versions of topological modular forms. These were constructed by Davies [Dav24, Dav25a].

Recall that the C_2-action on a complex vector bundle is given by complex conjugation, which carries over to the spectrum KU of complex K-theory. The C_2-action on the spectrum $\mathrm{TMF}_1(3)$ is induced by an underlying action as well: there is a natural $(\mathbb{Z}/n)^\times$-action on $\mathcal{M}_1(n)$ given by sending the point x of order n to $[k]x$ for $k \in (\mathbb{Z}/n)^\times$. It follows that both, KU and $\mathrm{TMF}_1(3)$, can be made genuinely equivariant with C_2-equivariant stable Adams operations.

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We will denote the C_2-spectra by $\mathbf{KR}$ and $\mathbf{TMF}_1(3)$. There is even more to say: in both cases the endomorphism $[-1]$ induces the endomorphism ψ^{-1} of spectra. If ψ^k is the endomorphism induced by $[k]$ on one of the above spectra, we even have coherently that $\psi^{-1} \circ \psi^k \simeq \psi^k \circ \psi^{-1}$ and $\psi^k \circ \psi^l \simeq \psi^{kl}$ making the ψ^k C_2-equivariant. These endomorphisms ψ^k are then called *stable Adams operations.*

The goal of this paper is to compute the C_2-equivariant stable Adams operations on the representation graded homotopy groups of these C_2-spectra. Since K-theory and topological modular forms are so closely related it is not surprising that the result and the strategy for the computations are essentially the same.

Theorem A (Definition 3.1)**.** *Let $k \in \mathbb{Z}$ and $x \in \pi^{C_2}_{a+b\sigma}\mathbf{KR}[\frac{1}{k}]$, where σ is the sign representation of C_2. Then*

$$\psi^k(x) = k^{\frac{a+b}{2}} x \in \pi^{C_2}_{a+b\sigma}\mathbf{KR}[\frac{1}{k}].$$

In particular, if x is torsion (necessarily 2-torsion), then $\psi^k(x) = x$.

Theorem B (Definition 3.2)**.** *Let $k \in \mathbb{Z}$ and $x \in \pi^{C_2}_{a+b\sigma}\mathbf{TMF}_1(3)[\frac{1}{k}]$, where σ is the sign representation of C_2. Then*

$$\psi^k(x) = k^{\frac{a+b}{2}} x \in \pi^{C_2}_{a+b\sigma}\mathbf{TMF}_1(3)[\frac{1}{k}].$$

In particular, if x is torsion (necessarily 2-torsion), then $\psi^k(x) = x$.

The computations build on the fact that stable Adams operations are maps of $\mathbb{E}_\infty$-rings, and it therefore suffices to compute them on the generators of the homotopy groups. There are three different kinds of generator, and for each we use a different technique. If the generator is

- coming from the sphere, we use $\mathbb{S}$-linearity.
- coming from the underlying nonequivariant spectrum, use the underlying computations.
- genuinely equivariant, use the homotopy fixed point spectra.

In Section 2, we will review relevant results about the computations of the $RO(C_2)$-graded homotopy groups of $\mathbf{KR}$ and $\mathbf{TMF}_1(3)$ as well as the construction of stable Adams operations. The main theorems are proven in Section 3. We will first deal with K-theory and then analogously handle topological modular forms.

Acknowledgments

The results of this article are part of my Masters' thesis at the University of Bonn. I thank my advisor Jack Davies for suggesting the topic, answering countless questions and giving invaluable advice on a draft of this paper. I am especially grateful to John Greenlees for sharing his LaTeX code for Figures 1 and 2. I also thank Emma Brink, Lennart Meier and Lucas Piessevaux for many helpful discussions about TMF, and Tom Bachmann and Lucas Piessevaux for reading drafts of this article at various stages. The author acknowledges support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Collaborative Research Centre TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124.

2 Recollections

The purpose of this section is to recall the necessary information about the homotopy groups of the C_2-spectra of Real K-theory and topological modular forms with $\Gamma_1(3)$ level structure. We will only give the parts needed for the computations in Section 3 and references for more details. We also quickly recall the existence of stable Adams operations on the relevant spectra.

Let σ be the sign representation of C_2. Recall that the representation ring is $RO(C_2) \simeq \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The $RO(C_2)$-graded homotopy groups of interest will have two generators which have a truly equivariant nature. The first one is the Euler class of σ, which is denoted as a_σ, cf. [HHR16, Definition 3.11]. Note that this class a_σ corresponds to the inclusion $S^0 \to S^\sigma$ and lives in degree $-\sigma$. The second class is the Euler-like class $u_{2\sigma}$ living in degree $2(1-\sigma)$, see [HHR16, Definition 3.12].

Real K-theory

Let us denote the genuine C_2-spectrum of Real K-theory of [Ati66] by $\mathbf{KR}$. Moreover, we denote the Bott element by $u \in \pi_2 \mathrm{KU}$. This has an equivariant lift $\overline{u} \in \pi^{C_2}_{1+\sigma}\mathbf{KR}$, i.e., the restriction morphism $\mathrm{res}^{C_2}_e \colon \pi^{C_2}_{1+\sigma}\mathbf{KR} \to \pi^e_2 \mathrm{KU}$ sends $\overline{u}$ to u. Computations of e.g. [Dug05, BG10, Gre18] show, that the $RO(C_2)$-graded homotopy groups $\pi^{C_2}_\bigstar \mathbf{KR}$ are generated by the elements:

$$a_\sigma, \quad u_{2\sigma}^{\pm 2}, \quad \overline{u} \quad \text{and} \quad v_0(m) \coloneqq 2u_{2\sigma}^m \text{ for } m \in \mathbb{Z} \tag{1}$$

such that $2v_0(2) = 4u_{2\sigma}^2$. Note that this notation is from the homotopy fixed point spectra sequence, and is not meant to imply that this is a product on $\pi_\bigstar$. The generators of Equation (1) are subject to several relations which we do not include here, as they are not relevant for our computations. Instead, we depict the homotopy groups of $\mathbf{kR}$, which give these of $\mathbf{KR}$ by inverting $\overline{u}$, in Figure 1. This figure was taken from [Gre18], but the notation was adapted to ours.

Nonequivariantly, stable Adams operations on the spectra KU of complex and KO of real topological K-theory were first introduced and completely computed by Adams [Ada62]. By [Lur18, Example 0.0.4] the C_2-action on KU is given by $[-1] \colon \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$ acting on the canonical orientation of $\hat{\mathbb{G}}_m$ over KU. Hence, by the discussion in [Dav25b, §6.4], this yields C_2-equivariant stable Adams operations ψ^k on $\mathbf{KR}[\frac{1}{k}]$ for $k \in \mathbb{Z}$, see also [DL25, Remark 6.7].

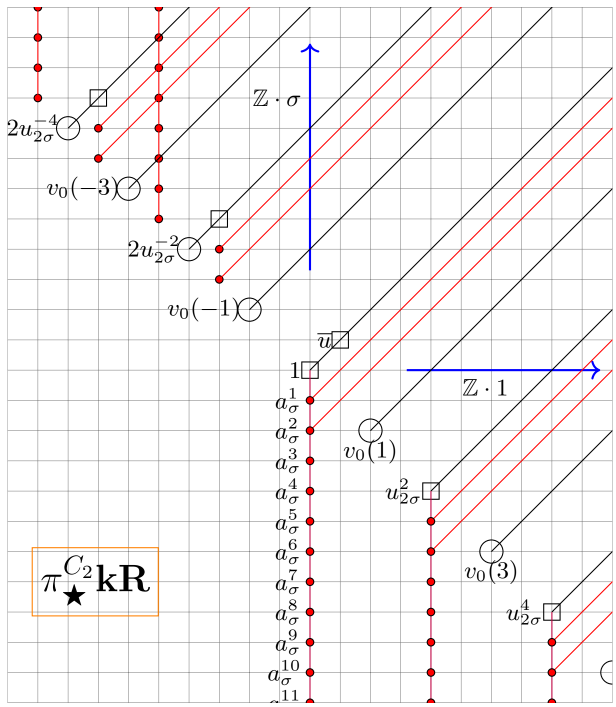

Figure 1: Black circles, squares and lines represent copies of $\mathbb{Z}$, red dots and lines represent copies of $\mathbb{F}_2$. For more details we refer to [Gre18, §3].

Topological modular forms of level structure $\Gamma_1(3)$

We denote the Borel C_2-spectrum of [HM17, Theorem 2.4] of topological modular form with $\Gamma_1(3)$ level structure by $\mathbf{TMF}_1(3)$. Its underlying nonequivariant spectrum is denoted by $\mathrm{TMF}_1(3)$ with homotopy groups $\pi_* \mathrm{TMF}_1(3) \cong \mathbb{Z}[\frac{1}{3}][a_1, a_3, \Delta^{-1}]$ [MR09, Corollary 3.3]. Here $|a_1| = 2$, $|a_3| = 6$ and $\Delta = a_3^3(a_1^3 - 27a_3)$. The equivariant lifts are denoted by $\overline{a}_1$ and $\overline{a}_3$ and lie in degree $1+\sigma$ and $3+3\sigma$, respectively. Moreover, Hill and Meier [HM17, §4.3] show that as C_2-spectra $\mathbf{tmf}_1(3)[\overline{\Delta}^{-1}] \cong \mathbf{TMF}_1(3)$, where $\overline{\Delta} = \overline{a}_3^3(\overline{a}_1^3 - 27\overline{a}_3)$ is the equivariant lift of Δ.

Combining [HM17, Theorem 4.15] and [GM17, §13.B] one obtains that $\pi_\bigstar^{C_2}\mathbf{TMF}_1(3)$ is generated by the following classes:

$$a_\sigma, \quad \overline{a}_1, \quad \overline{a}_3, \quad u_{2\sigma}^4, \quad \overline{a}_1(1) \coloneqq \overline{a}_1 u_{2\sigma}^2 \text{ and } \quad v_0(m) \coloneqq 2u_{2\sigma}^m \text{ for } m \in \mathbb{Z}. \quad (2)$$

The same remarks as in the K-theory case hold. A picture of the homotopy groups of $\mathbf{tmf}_1(3)$, which give these of $\mathbf{TMF}_1(3)$ by inverting $\overline{\Delta}$, is Figure 2, which is taken from [GM17, §13.B].

The C_2-spectrum $\mathbf{TMF}_1(3)$ has C_2-equivariant stable Adams operations

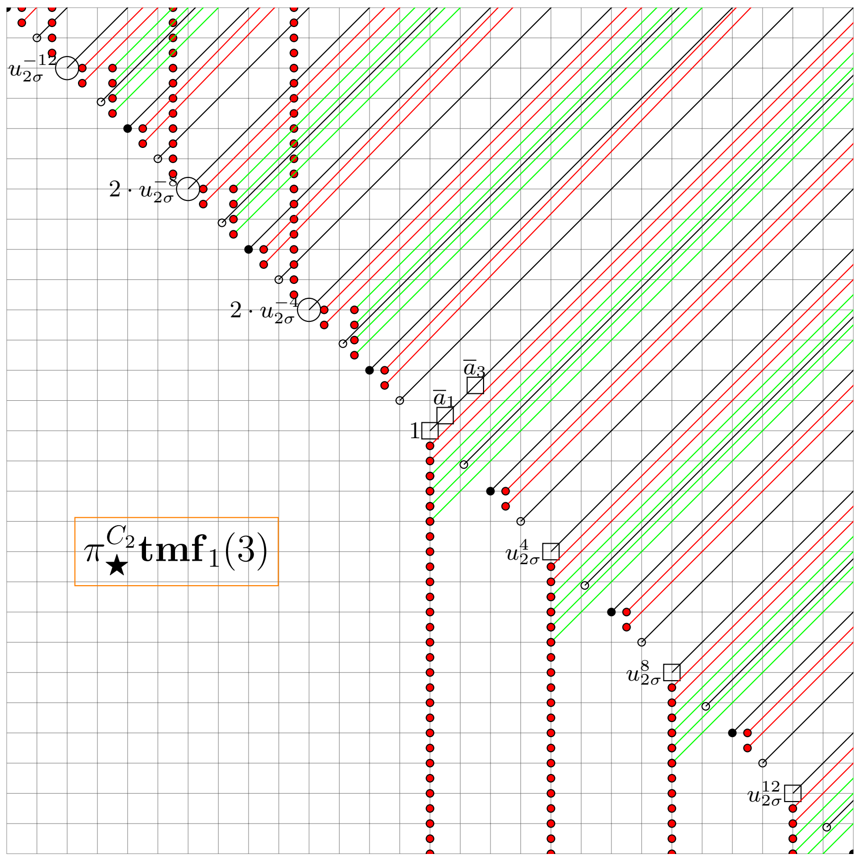

Figure 2: Black circles, squares and lines represent copies of $\mathbb{Z}$; red dots represent copies of $\mathbb{F}_2$, red lines mean a copy of $\mathbb{F}_2[\overline{a}_1, \overline{a}_3]$ and green lines mean a copy of $\mathbb{F}_2[\overline{a}_3]$. For more details we refer to [GM17, §13.B]. The axes are the same as in Figure 1

by [Dav24, Theorem C and Proposition 2.15] and the remarks directly after these. Let us make this more precise. By [Dav24, Theorem A], there is a functor $\mathcal{O}^{\mathrm{top}}\colon \mathrm{Isog}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp})$. Here, Isog is the 2-category whose objects are stacks X equipped with an étale map to the moduli stack of elliptic curves $\mathcal{M}_{ell}$, which classifies an elliptic curve E over X, and morphisms are pairs (f,φ) of a map of stacks $f\colon X' \to X$ and an isogeny of elliptic curves $\varphi\colon E' \to f^*E$ of invertible degree. The $\mathbb{E}_\infty$-ring $\mathrm{TMF}_1(3)$ is the image of the moduli stack $\mathcal{M}_1(3)$ of elliptic curves with $\Gamma_1(3)$-level structure [MR09, §2] with C_2-action given by the automorphism $(id,[-1])$ of $(\mathcal{M}_1(3), E_1(3))$, where $E_1(3)$ is the universal elliptic curve living over $\mathcal{M}_1(3)$. By [Dav24, Diagram (2.2)], we define ψ^n on $\mathrm{TMF}_1(3)[\frac{1}{n}]$ using the endomorphism $(id,[n])$ of $(M_1(3), E_1(3))$ and applying $\mathcal{O}^{\mathrm{top}}$. In the 2-category Isog, there is a natural 2-morphism between $(id,[n])\circ(id,[-1]) \simeq (id,[-1])\circ(id,[n])$ which shows that the maps $(id,[n])$ are C_2-equivariant in Isog. Applying $\mathcal{O}^{\mathrm{top}}$ then shows that the Adams operations ψ^n on $\mathrm{TMF}_1(3)[\frac{1}{n}]$ are C_2-equivariant in CAlg(Sp).

3 Computations of Adams operations

In this section, we compute the stable Adams operations on the $RO(C_2)$-graded homotopy groups of $\mathbf{KR}$ and $\mathbf{TMF}_1(3)$.

Real K-theory

Here we compute the effect of stable Adams operations on the $RO(C_2)$-graded homotopy groups of the C_2-spectrum $\mathbf{KR}$. There is a second C_2-spectrum related to topological complex K-theory, which is denoted by KU_{C_2} and defined by Segal [Seg68]. Note that $\mathbf{KR} \neq \mathrm{KU}_{C_2}$, but there is a map $\mathrm{KU}_{C_2} \to \mathbf{KR}$. Balderrama [Bal22, Lemma 2.2.2] computed the stable Adams operations for KU_{C_2}, and even for more general KU_G. This computation and the above map presumably could be used to recover the next result, but we will use the same techniques as for Definition 3.2.

Theorem 3.1. *Let $k \in \mathbb{Z}$ and $x \in \pi^{C_2}_{a+b\sigma}\mathbf{KR}[\frac{1}{k}]$. Then $\psi^k(x) = k^{\frac{a+b}{2}}x \in \pi^{C_2}_{a+b\sigma}\mathbf{KR}[\frac{1}{k}]$. In particular, if x is torsion (necessarily 2-torsion), then $\psi^k(x) = x$.*

Proof. We will implicitly invert k everywhere in the proof. As stable Adams operations are multiplicative and additive it suffices to show the statement for the generators a_σ, $\overline{u}$, $u^2_{2\sigma}$ and $v_0(m)$ of Equation (1).

Recall that $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbf{KR}$ is the image of $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbb{S}$ under the unit $\mathbb{S} \to \mathbf{KR}$ and that the stable Adams operations ψ^k are $\mathbb{S}$-linear with $\psi^k(1) = 1$, meaning they commute with the unit:

$$\psi^k(a_\sigma) = a_\sigma\psi^k(1) = a_\sigma.$$

The C_2-spectrum $\mathbf{KR}$ is strongly even by [GM17, Examples 4.13], i.e. the morphism $\mathrm{res}^{C_2}_e\colon \pi^{C_2}_{n(1+\sigma)}\mathbf{KR} \to \pi^e_{2n}\mathbf{KR} = \pi_{2n}\,\mathrm{KU}$ is an isomorphism for all $n \in$

$\mathbb{Z}$. Since restriction maps commute with stable Adams operations by naturality, the result for $\overline{u}$ follows by the classical computation in KU, cf. [Ada62, Corollary 5.2].

Lastly, we compute the Adams operations for the classes $u_{2\sigma}^2$ and $v_0(n)$. For this we use the well known equivalence $\mathbf{KR}^{hC_2} \simeq \mathrm{KO}$, see [Ati66]. By Figure 1, we have isomorphisms

$$\overline{u}^n \colon \pi^{C_2}_{n(1-\sigma)}\mathbf{KR} \xrightarrow{\simeq} \pi^{C_2}_{2n}\mathbf{KR} = \pi_{2n}\,\mathrm{KO} \tag{3}$$

given by multiplication. By [Ada62, Corollary 5.2], the ψ^k-action on $\pi_*\,\mathrm{KO}$ is given by multiplication by k^n in degree $2n$. Now compute

$$k^4\overline{u}^4u_{2\sigma}^2 = \psi^k(\overline{u}^4u_{2\sigma}^2) = \psi^k(\overline{u}^4)\psi^k(u_{2\sigma}^2) = k^4\overline{u}^4 \cdot \psi^k(u_{2\sigma}^2) \in \pi_8^{C_2}\mathbf{KR} \simeq \mathbb{Z}.$$

Since k is invertible and Equation (3) is an isomorphism, this implies that $u_{2\sigma}^2 = \psi^k(u_{2\sigma}^2)$. The same argument applied to $\overline{u}^{2m} \cdot v_0(m)$ also shows that $\psi^k(v_0(m)) = v_0(m)$, as desired. □

Topological modular forms of level structure $\Gamma_1(3)$

The strategy to compute the stable Adams operations on $\pi^{C_2}_\bigstar\mathbf{TMF}_1(3)$ is precisely the same as for Real K-theory.

Theorem 3.2. *Let $k \in \mathbb{Z}$ and $x \in \pi^{C_2}_{a+b\sigma}\mathbf{TMF}_1(3)[\frac{1}{k}]$. Then $\psi^k(x) = k^{\frac{a+b}{2}}x \in \pi^{C_2}_{a+b\sigma}\mathbf{TMF}_1(3)[\frac{1}{k}]$. In particular, if x is torsion (necessarily 2-torsion), then $\psi^k(x) = x$.*

Proof. Again, k will be implicitly inverted throughout the proof. As stable Adams operations are multiplicative it suffices to show the statement for the generators a_σ, $\overline{a}_1$, $\overline{a}_3$, $u_{2\sigma}^4$, $v_0(m)$ and $\overline{a}_1(1)$ of [HM17, Theorem 4.15].

Recall that $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbf{TMF}_1(3)$ is the image of $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbb{S}$ under the unit $\mathbb{S} \to \mathbf{TMF}_1(3)$ and that the stable Adams operations ψ^k are $\mathbb{S}$-linear with $\psi^k(1) = 1$, meaning they commute with the unit:

$$\psi^k(a_\sigma) = a_\sigma\psi^k(1) = a_\sigma.$$

By [HM17, Before Proposition 4.23], $\mathbf{TMF}_1(3)$ is strongly even, i.e. for all $n \in \mathbb{Z}$ the morphism

$$\mathrm{res}^{C_2}_e \colon \pi^{C_2}_{n(1+\sigma)}\mathbf{TMF}_1(3) \to \pi^e_{n(1+\sigma)}\mathbf{TMF}_1(3) \cong \pi^e_{2n}\mathbf{TMF}_1(3) \cong \pi_{2n}\,\mathrm{TMF}_1(3)$$

is an isomorphism. Combining [MR09, Corollary 3.3] and [Dav24, Theorem B] yields the result for $\overline{a}_1$ and $\overline{a}_3$.

By étale descent there is an equivalence $\mathbf{TMF}_1(3)^{hC_2} \simeq \mathrm{TMF}_0(3)$, see [MR09, §2], hence, using Figure 2, we have isomorphisms

$$\overline{a}_1^n \colon \pi^{C_2}_{n(1-\sigma)}\mathbf{TMF}_1(3) \xrightarrow{\simeq} \pi^{C_2}_{2n}\mathbf{TMF}_1(3) = \pi_{2n}\,\mathrm{TMF}_0(3) \tag{4}$$

given by multiplication. Using [Dav25b, Proposition 6.18] we obtain that the ψ^k-action on $\pi_* \operatorname{TMF}_0(3)$ is given by multiplication by k^n in degree $2n$. Now compute

$$k^8\overline{a}_1^8 u_{2\sigma}^4 = \psi^k(\overline{a}_1^8 u_{2\sigma}^4) = \psi^k(\overline{a}_1^8)\psi^k(u_{2\sigma}^4) = k^8\overline{a}_1^8 \cdot \psi^k(u_{2\sigma}^4) \in \pi_{16}^{C_2}\mathbf{TMF}_1(3) \simeq \mathbb{Z}.$$

Since k is invertible and Equation (4) is an isomorphism, this implies that $u_{2\sigma}^4 = \psi^k(u_{2\sigma}^4)$. The same argument applied to $\overline{a}_1^{2m} \cdot v_0(m)$ and $\overline{a}_1^3 \cdot \overline{a}_1(1)$ also shows that $\psi^k(v_0(m)) = v_0(m)$ and $\psi^k(\overline{a}_1(1)) = \overline{a}_1(1)$, respectively. This ends the proof. □

Remark 3.3. Note that the computation of $\psi^k(\overline{a}_1(1))$ can also be done by applying the same argument as above to $\overline{a}_3 \cdot \overline{a}_1(1)$.

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